

# NEWTON'S METHOD AND BAKER DOMAINS

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ABSTRACT. We show that there exists an entire function  $f$  without zeros for which the associated Newton function  $N_f(z) = z - f(z)/f'(z)$  is a transcendental meromorphic function without Baker domains. We also show that there exists an entire function  $f$  with exactly one zero for which the complement of the immediate attracting basin has at least two components and contains no invariant Baker domains of  $N_f$ . The second result answers a question of J. Rückert and D. Schleicher while the first one gives a partial answer to a question of X. Buff.

## 1. INTRODUCTION AND RESULTS

Newton's method for finding the zeros of an entire  $f$  consists of iterating the meromorphic function

$$N_f(z) := z - \frac{f(z)}{f'(z)},$$

see [1] for an introduction to the iteration theory of meromorphic functions, including a section on Newton's method. If  $\xi$  is a zero of  $f$ , then there exists an  $N_f$ -invariant component  $U$  of the Fatou set of  $N_f$  containing  $\xi$  in which the iterates  $N_f^k$  of  $N_f$  converge to  $\xi$  as  $k \rightarrow \infty$ . This domain  $U$  is called the *immediate basin* of  $\xi$ .

There may also be  $N_f$ -invariant components of the Fatou set of  $N_f$  in which the iterates of  $N_f$  tend to  $\infty$ . We call such an  $N_f$ -invariant domain a *virtual immediate basin*. (This is in slight deviation from [4, 10] where the definition of a virtual immediate basin additionally includes the existence of an "absorbing set"; cf. the remark in §3.3.) It was suggested by Douady that the existence of virtual immediate basins is related to 0 being an asymptotic value of  $f$ . This relationship was investigated in [2, 4]. While it was shown in [2] that in general the existence of a virtual immediate basin does not imply that 0 is an asymptotic value of  $f$ , this conclusion was shown to be true under suitable additional hypotheses in [4]. It was also shown in [4] that if  $f$  has a logarithmic singularity over 0, then  $N_f$  has a virtual immediate basin.

If  $f$  has the form  $f = Pe^Q$  where  $P$  and  $Q$  are polynomials, with  $Q$  nonconstant, then the Newton function  $N_f$  is rational,  $\infty$  is a parabolic fixed point of  $N_f$  and the associated parabolic domains are virtual immediate basins. If  $f$  does not have the above form, then  $N_f$  is transcendental. An invariant Fatou component where the iterates of  $N_f$  tend to  $\infty$  is then called an *invariant Baker domain*. So except in the case where  $f = Pe^Q$  a virtual immediate basin is an invariant Baker domain.

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If  $f$  has no zeros, then  $f$  has the asymptotic value 0 by Iversen's theorem [8, p. 292]. This suggests that there could always be virtual immediate basins if there are no zeros. We show that this is not the case in general.

**Theorem 1.** *There exists an entire function  $f$  without zeros for which  $N_f$  is a transcendental meromorphic function without invariant Baker domains.*

The following corollary is obvious.

**Corollary.** *There exists a transcendental meromorphic function without fixed points and without invariant Baker domains.*

This is a partial answer to a question of Buff who had asked whether there exists a transcendental *entire* function without fixed points and without invariant Baker domains.

Rückert and Schleicher [10] have shown that if  $f$  is a polynomial and if  $U$  is the immediate basin of a zero, then each component of  $\mathbb{C} \setminus U$  contains the basin of another zero. They deduce this result from a more general result dealing with the case that  $f$  is entire but not necessarily a polynomial. To state this result, let again  $U$  be the immediate basin of a zero  $\xi$  of  $f$  and suppose that there are two  $N_f$ -invariant curves  $\Gamma_1$  and  $\Gamma_2$  which connect  $\xi$  to  $\infty$  in  $U \cup \{\infty\}$ , which intersect only in  $\xi$  and  $\infty$  and which are not homotopic (with fixed endpoints) in  $U \cup \{\infty\}$ . Let  $\tilde{V}$  be a component of  $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_2)$ . With these notations their main result [10, Theorem 5.1] takes the following form.

**Theorem (Rückert, Schleicher).** *If no point in  $\hat{\mathbb{C}}$  has infinitely many preimages within  $\tilde{V}$ , then the set  $V := \tilde{V} \setminus U$  contains an immediate basin or a virtual immediate basin of  $N_f$ .*

Rückert and Schleicher raise the question whether the hypothesis that no point in  $\hat{\mathbb{C}}$  has infinitely many preimages within  $\tilde{V}$  is necessary. We show that this is indeed the case.

**Theorem 2.** *There exists an entire function  $g$  with exactly one zero at 0 such that the immediate basin of 0 contains  $\mathbb{R}$ , but  $N_g$  has no virtual immediate basin.*

The functions  $f$  and  $g$  in Theorems 1 and 2 can be given explicitly. Let  $(r_k)$  be a sequence of real numbers tending to  $\infty$  and let  $(n_k)$  be a sequence of positive integers satisfying  $n_k \geq k$  for all  $k \in \mathbb{N}$ . Then

$$(1.1) \quad h(z) := \prod_{k=1}^{\infty} \left( 1 + \left( \frac{z}{r_k} \right)^{n_k} \right)$$

defines an entire function  $h$ . We shall show that if

$$(1.2) \quad r_k \geq 2r_{k-1} \geq 2, \quad n_k \geq \sum_{j=1}^{k-1} n_j \quad \text{and} \quad n_k \geq r_k^{4n_{k-1}}$$

for  $k \geq 2$ , then the functions

$$(1.3) \quad f(z) := \exp \left( \int_0^z h(t) dt \right)$$

and

$$(1.4) \quad g(z) := z \exp \left( \int_0^z \frac{h(t) - 1}{t} dt \right)$$

satisfy the conclusions of Theorems 1 and 2, respectively.

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## 2. PROOFS OF THEOREM 1 AND 2

We denote the open disk of radius  $r$  around a point  $a \in \mathbb{C}$  by  $D(a, r)$ . The hyperbolic metric in a plane domain  $U$  is denoted by  $\lambda_U$ . By  $\text{dist}_U(z, A)$  we denote the hyperbolic distance between a point  $z$  and a set  $A$ . We shall make use of the fact that if  $A \subset U$  and if  $z_0$  is a point which is in the boundary of  $U$  but not in the closure of  $A$ , then  $\text{dist}_U(z, A) \rightarrow \infty$  as  $z \rightarrow z_0$ ,  $z \in U$ . In particular, we have

$$(2.1) \quad \lim_{z \rightarrow \infty} \text{dist}_{\mathbb{C} \setminus \{0,1\}} \left( z, D\left(\frac{1}{2}, \frac{1}{2}\right) \right) = \infty.$$

To estimate the growth of  $h$  we note that (1.2) implies that if  $k \geq 3$ , then  $r_k \geq 2^{k-1} \geq 4$  and  $n_{k-1} \geq r_{k-1}^{4n_{k-2}} \geq 2^{(k-1)4n_{k-2}} \geq k+1$ . For  $|z| = r_k$  where  $k \geq 3$  we thus obtain

$$\begin{aligned} \log |h(z)| &\leq \sum_{j=1}^{k-1} \log \left( 1 + \left| \frac{r_k}{r_j} \right|^{n_j} \right) + \log 2 + \sum_{j=k+1}^{\infty} \log \left( 1 + \left| \frac{r_k}{r_j} \right|^{n_j} \right) \\ &\leq \sum_{j=1}^{k-1} \log (1 + r_k^{n_j}) + \log 2 + \sum_{j=k+1}^{\infty} \left| \frac{r_k}{r_j} \right|^{n_j} \\ &\leq \sum_{j=1}^{k-1} \log (2r_k^{n_j}) + \log 2 + \sum_{j=k+1}^{\infty} 2^{-n_j} \\ &\leq k \log 2 + \left( \sum_{j=1}^{k-1} n_j \right) \log r_k + 1 \\ &\leq (k + 2n_{k-1} + 1) \log r_k \\ &\leq 3n_{k-1} \log r_k. \end{aligned}$$

Hence

$$(2.2) \quad |h(z)| \leq r_k^{3n_{k-1}}.$$

for  $|z| = r_k$  and  $k \geq 3$ .

*Proof of Theorem 1.* Let  $h$  and  $f$  be defined by (1.1) and (1.3) so that

$$N_f(z) = z - \frac{1}{h(z)}.$$

Suppose that  $N_f$  has an invariant Baker domain  $U$ . Take  $z_0 \in U$  and connect  $z_0$  by a curve  $\gamma_0$  in  $U$  to  $N_f(z_0)$ . Then  $\gamma := \bigcup_{j=0}^{\infty} N_f^j(\gamma_0)$  is a curve in  $U$  which connects  $z_0$  to  $\infty$ . By compactness, there exists  $K \geq 0$  such that  $\lambda_U(z, N_f(z)) \leq K$  for all

$z \in \gamma_0$ . Since every  $z \in \gamma$  has the form  $z = N_f^j(\zeta)$  for some  $\zeta \in \gamma_0$  and some  $j \geq 0$  and since the holomorphic self-map  $N_f$  of  $U$  does not increase hyperbolic distances this implies that

$$(2.3) \quad \lambda_U(z, N_f(z)) \leq K \quad \text{for } z \in \gamma.$$

For large  $k$  the curve  $\gamma$  intersects the circle  $\{z : |z| = r_k\}$ . Let  $z_k$  be a point of intersection. Define

$$P_k := \{r_k e^{(2\nu+1)\pi i/n_k} : 0 \leq \nu \leq n_k - 1\}.$$

The  $n_k$  points of  $P_k$  are zeros of  $h$  and hence poles of  $N_f$ . Thus  $P_k \cap U = \emptyset$  for all  $k \in \mathbb{N}$ . For  $k \geq 2$  we have  $n_k \geq r_k^4 \geq 16$  so that  $P_k$  contains more than one point. Let  $a_k, b_k$  the points of  $P_k$  which are closest to  $z_k$ . Then

$$(2.4) \quad |a_k - b_k| = |e^{2\pi i/n_k} - 1| \leq \frac{4\pi}{n_k}$$

and

$$(2.5) \quad z_k \in D\left(\frac{1}{2}(a_k + b_k), \frac{1}{2}|a_k - b_k|\right).$$

Define  $L_k : \mathbb{C} \setminus \{a_k, b_k\} \rightarrow \mathbb{C} \setminus \{0, 1\}$  by  $L_k(z) = (z - a_k)/(b_k - a_k)$ . Then

$$(2.6) \quad \lambda_{\mathbb{C} \setminus \{0,1\}}(L_k(z_k), L_k(N_f(z_k))) = \lambda_{\mathbb{C} \setminus \{a_k, b_k\}}(z_k, N_f(z_k)) \leq \lambda_U(z_k, N_f(z_k)) \leq K$$

by (2.3). By (2.5) we have  $L_k(z_k) \in D(\frac{1}{2}, \frac{1}{2})$ . On the other hand, (2.2), (2.4) and (1.2) imply that

$$\begin{aligned} |L_k(N_f(z_k))| &\geq |L_k(N_f(z_k)) - L_k(z_k)| - |L_k(z_k)| \\ &= \frac{|N_f(z_k) - z_k|}{|a_k - b_k|} - |L_k(z_k)| \\ &= \frac{1}{|h(z_k)(a_k - b_k)|} - |L_k(z_k)| \\ &\geq \frac{n_k}{4\pi r_k^{3n_k-1}} - 1 \\ &\geq \frac{r_k^{n_k-1}}{4\pi} - 1 \end{aligned}$$

and thus  $|L_k(N_f(z_k))| \rightarrow \infty$  as  $k \rightarrow \infty$ . Combining this with (2.1) we see that  $\lambda_{\mathbb{C} \setminus \{0,1\}}(L_k(z_k), L_k(N_f(z_k))) \rightarrow \infty$  as  $k \rightarrow \infty$ , contradicting (2.6).  $\square$

*Proof of Theorem 2.* Let  $h$  and  $g$  be defined by (1.1) and (1.4) so that

$$N_g(z) = z - \frac{z}{h(z)} = z \left(1 - \frac{1}{h(z)}\right).$$

The proof that  $N_g$  has no Baker domains proceeds exactly as the proof of Theorem 1. (We only obtain

$$|L_k(N_f(z_k))| \geq \frac{r_k^{n_k-1}}{4\pi} - 1,$$

but this still gives a contradiction.)

Since  $h$  is real on the real axis, the same holds for  $N_g$ , and since  $h(x) > 1$  for all  $x \in \mathbb{R} \setminus \{0\}$  we see that  $|N_g(x)| < |x|$  for all  $x \in \mathbb{R} \setminus \{0\}$ . This implies that

$N_g^k(x) \rightarrow 0$  as  $k \rightarrow \infty$ , for all  $x \in \mathbb{R}$ . Hence  $\mathbb{R}$  is contained in the immediate basin of 0.  $\square$

### 3. REMARKS

**1.** It follows from the result of Buff and Rückert [4] already mentioned in the introduction that the function  $f$  of Theorem 1 has no logarithmic singularity over 0. Another example of an entire function without zeros and with no logarithmic singularity over 0 was given in [3].

**2.** The invariant components of the Fatou set of a meromorphic function can be classified; see [1]. For functions without fixed points there are only two possible types of invariant components: Baker domains and Herman rings. Fagella, Jarque and Taixes [6] have shown that a meromorphic function without fixed points does not have Herman rings. This implies that for a function  $f$  satisfying the conclusion of Theorem 1 the Fatou set of  $N_f$  has no invariant component at all. Probably there also exist entire functions  $f$  for which the Fatou set of  $N_f$  is empty.

**3.** It was shown by Przytycki [9] that if  $f$  is a polynomial, then the immediate basin of each zero is simply connected. Shishikura [11] showed that in fact the Julia set of  $N_f$  is connected; that is, all Fatou components of  $N_f$  are simply connected. It is not known whether this last result also holds if  $f$  is an entire transcendental function, but Mayer and Schleicher [7] have shown that immediate basins are simply connected. Fagella, Jarque and Taixes [5, 6] have extended this result by showing that immediate attracting and parabolic basins (of any period) are simply connected and that preimages of simply connected Fatou components of  $N_f$  are simply connected. However, it remains open whether invariant Baker domains of  $N_f$  are necessarily simply connected. If this is true, then our definition of virtual immediate basins coincides with that given in [4, 10] since then the additional condition on the existence of an absorbing set is always satisfied; cf. the discussion in [4, 10].

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